Directed Minors III.
Directed Linked Decompositions

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Abstract

Thomas proved that every undirected graph admits a linked tree decomposition of width equal to its treewidth. In this paper, we generalize Thomas's theorem to digraphs. We prove that every digraph G admits a linked directed path decomposition and a linked DAG decomposition of width equal to its directed pathwidth and DAG-width respectively.

Keywords: DAG-decomposition, DAG-width, Directed path decomposition, Directed pathwidth, Menger's theorem.

1 Introduction

Let G be an undirected graph and let $\mathsf{tw(G)}$ denote its treewidth. Robertson and Seymour [RS90] proved that every undirected graph admits a linked tree decomposition of width $< 3.2^{\mathsf{tw(G)}}$. This theorem is a crucial technical tool for proving that every set of bounded treewidth graphs is well-quasi-ordered. Thomas [Tho90] improved this theorem with the best possible bound i.e., every undirected graph admits a linked tree decomposition of width equal to its treewidth (see [BD02] for an alternate proof). An analogous theorem for branch-width was proved by Geelen, Gerards and Whittle [GGW02]. They used this result to prove that all matroids representable over a fixed finite field and with bounded branch-width are well-quasi-ordered under minors. Kim and Seymour [KS12] proved that every semi-complete digraph admits a linked directed path decomposition of width equal to its directed pathwidth. They used this result to show that all semi-complete digraphs are well-quasi-ordered under "strong" minors[1].

Tree decomposition (resp. path decomposition) of an undirected graph G is a collection of subsets of vertices (called *bags*) attached to an underlying tree (resp. path). These bags correspond to a certain "separators" of G. Linked decompositions are a kind of "canonical" decompositions based on "minimum" separators and hence satisfying a "menger-like" property. Thomas's theorem

[1] A digraph H is a "strong" minor of a digraph G if H can be obtained from a subdigraph of G by repeatedly contracting a strongly-connected subdigraph to a vertex.

states that this property can be achieved in the optimal decomposition without increasing the width of the bags.

Directed path decompositions and DAG decompositions are based on a notion of *guarding*, which is a natural generalization of undirected *separators* to digraphs (see Section 1.2). Hence, directed pathwidth and DAG-width are naturally suited to study *directed* linked decompositions.

Directed pathwidth was introduced by Reed, Seymour and Thomas (see [Bar06]). Berwanger et al. [BDHK06] and independently Obdržálek [Obd06] introduced DAG-width. For an undirected graph G, let $\mathsf{pw}(G)$ and $\mathsf{tw}(G)$ denote its pathwidth and treewidth respectively. For a digraph G, let $\mathsf{dpw}(G)$ and $\mathsf{dgw}(G)$ denote its directed pathwidth and DAG-width respectively. The following proposition further emphasizes the "naturalness" of directed pathwidth and DAG-width.

Proposition 1. For an *undirected* graph G, let $\overset{\leftrightarrow}{G}$ be the *digraph* obtained by replacing each edge $\{u, v\}$ of G by two directed edges (u, v) and (v, u), then: (i) $\mathsf{dpw}(\overset{\leftrightarrow}{G}) = \mathsf{pw}(G)$ [Bar06, Lemma 1] and (ii) $\mathsf{dgw}(\overset{\leftrightarrow}{G}) = \mathsf{tw}(G) + 1$ [BDHK06, Proposition 5.2].

In this paper, we generalize Thomas's theorem to digraphs. We prove that every digraph G admits a linked directed path decomposition and a linked DAG decomposition of width equal to its directed pathwidth and DAG-width respectively. Similar to [Tho90, BD02], we prove a stronger result using "lean" decompositions (see **(DPW-4)**, **(DPW-5)**, **(DGW-4)** and **(DGW-5)** for definitions of linked and lean decompositions). Our main theorems are Theorem 7 and Theorem 14. Our techniques generalize the alternate proof of Thomas's theorem given by Bellenbaum and Diestel [BD02].

1.1 Basic Notation

We use standard graph theory notation and terminology (see [Die05]). All digraphs are finite and simple (i.e. no self loops and no multiple arcs). For a digraph G, we write $V(G)$ for its vertex set and $E(G)$ for its arc set. For $S \subseteq V(G)$ we write $G[S]$ for the subdigraph induced by S, and $G \setminus S$ for the subdigraph induced by $V(G) - S$.

We use the term DAG when referring to directed acyclic graphs. A node is a *root* if it has no incoming arcs. Let T be a DAG. For two *distinct* nodes i and j of T, we write $i \prec_T j$ if there is a directed walk in T with first node i and last node j. For convenience, we write $i \prec j$ whenever T is clear from the context. For nodes i and j of T, we write $i \preceq j$ if either $i = j$ or $i \prec j$. We define $T_{\succeq v} = T[\{x \mid x \succeq v\}]$ and $T_{\succ v} = T[\{x \mid x \succ v\}]$. For $t_1, t_2 \in V(T)$, let $d_T(t_1, t_2)$ denote the shortest directed distance from t_1 to t_2. Whenever $t_1 \preceq t \preceq t_2$ we say $t \in [t_1, t_2]$.

Let $\mathcal{X} = (X_i)_{i \in V(T)}$ be a family of finite sets called *node bags*, which associates each node i of T to a node bag X_i. We write $X_{\succeq i}$ to denote $\bigcup\limits_{j \succeq i} X_j$.

1.2 Separation and Guarding

Definition 2. [Separation] Let G be a digraph and $A, B \subseteq V(G)$. We say (A, B) is a *separation of G of order s* if:

- $A \cup B = V(G)$,
- $|A \cap B| = s$, and

- there is no edge *from* $A \setminus B$ *to* $B \setminus A$.

For $X, Y \subseteq V(G)$, we say that (A, B) *separates* (X, Y) if $X \subseteq A$ and $Y \subseteq B$. Alternately we say that $A \cap B$ separates (X, Y).

Theorem 3. (Menger's Theorem [Men27]) For a digraph G, two subsets $X, Y \subseteq V(G)$ and an integer $k \geq 1$, exactly one of the following holds:

- there are k vertex-disjoint directed paths from X to Y.
- there is a separation (A, B) of G of order $< k$ with $X \subseteq A$ and $Y \subseteq B$.

Directed pathwidth and DAG-width are based on the following notion of *guarding*:

Definition 4. [Guarding] Let G be a digraph and $W, X \subseteq V(G)$. We say X *guards* W if $W \cap X = \emptyset$, and for all $(u, v) \in E(G)$, if $u \in W$ then $v \in W \cup X$.

In other words, X guards W means that there is no directed path in $G \setminus X$ that starts from W and leaves W.

2 Linked directed path decomposition

Definition 5. [Directed path decomposition and Directed pathwidth [Bar06]] A *Directed path decomposition* of a digraph G is a sequence $X_1, X_2, \ldots, X_r$ of subsets (node bags) of $V(G)$, such that:

- $\bigcup_{1 \leq i \leq r} X_i = V(G)$. (DPW-1)
- For all $i, j, k \in [r]$, if $i \leq j \leq k$, then $X_i \cap X_k \subseteq X_j$. (DPW-2)
- For all arcs $(u, v) \in E(G)$, there exist i, j with $1 \leq i \leq j \leq r$ such that $u \in X_i$ and $v \in X_j$. (DPW-3)

The width of a directed path decomposition $\mathcal{X} = (X_i)_{i \in [r]}$ is defined as $\max\{|X_i| : 1 \leq i \leq r\} - 1$. The *directed pathwidth* of G, denoted by $\mathsf{dpw}(G)$, is the minimum width over all possible directed path decompositions of G.

(DPW-2) can be replaced by the following equivalent statement:

- For any $v \in V(G)$, $\{i : X_i \cap v \neq \emptyset, 1 \leq i \leq r\}$ is an integer interval. (DPW-2')

(DPW-3) can be replaced by the following equivalent statement:

- For any i with $1 < i < r$, there is no edge *from* $\bigcup\limits_{i+1 \leq j \leq r} X_j$ *to* $\bigcup\limits_{1 \leq j \leq i-1} X_j$ in $G \setminus X_i$. (DPW-3')

A directed path decomposition is called *linked* if it satisfies the following condition:

- Given any $k > 0$ and $t_1, t_2 \in [r]$ such that $t_1 \leq t_2$, either G contains k vertex-disjoint directed paths *from* X_{t_2} *to* X_{t_1} or there exists $i \in [t_1, t_2]$ such that $|X_i| < k$. (DPW-4)

A directed path decomposition is called *lean* if it satisfies the following condition:

- Given any $k > 0$ and $t_1, t_2 \in [r]$ such that $t_1 \le t_2$,
  and subsets $Z_1 \subseteq X_{t_1}$, and $Z_2 \subseteq X_{t_2}$ such that $|Z_1| = |Z_2| =: k$,
  either G contains k vertex-disjoint directed paths *from* Z_2 *to* Z_1
  or there exists $i \in [t_1, t_2 - 1]$ such that $|X_i \cap X_{i+1}| < k$. (DPW-5)

Lemma 6. (*Path meeting lemma*) For $l \ge 2$, let $P = v_1 v_2 \dots v_l$ be a directed path in a digraph G. Let $\mathcal{X} = (X_i)_{i\in[r]}$ be a directed path decomposition of G such that $v_l \in X_a$ and $v_1 \in X_b$ and $a < b$. Then $V(P) \cap X_i \ne \emptyset$ for all $a \le i \le b$.

Proof. We may assume $b - a \ge 2$. Let $V(P) \cap X_j = \emptyset$ for some c such that $a < c < b$. Let $\mathcal{X}_{left} = \bigcup_{i=1}^{c-1} X_i$ and $\mathcal{X}_{right} = \bigcup_{i=c+1}^{r} X_i$. By (DPW-2'), for any $x \in V(P)$, x is in $\mathcal{X}_{left}$ or $\mathcal{X}_{right}$, but not in both. Since $v_l \in \mathcal{X}_{left}$ and $v_1 \in \mathcal{X}_{right}$, there is an edge $(v_j, v_{j+1}) \in E(P)$ such that $v_j \in \mathcal{X}_{right}$ and $v_{j+1} \in \mathcal{X}_{left}$. This violates (DPW-3'). □

Theorem 7. Every digraph G has a directed path decomposition of width $\mathsf{dpw}(G)$ that satisfies (DPW-1) − (DPW-5).

Proof. Note that (DPW-5) generalizes[2] (DPW-4). Let the *fatness* of a directed path decomposition be the n-tuple $(f_n, f_{n-1}, \dots, f_0)$, where $f_i = |\{j \ : \ |X_j| = i\}|$ and $n := |V(G)|$. Let $\mathcal{X} = (X_i)_{i\in[r]}$ be a directed path decomposition of lexicographically minimal fatness. It is easy to see that $\mathcal{X}$ has width $\mathsf{dpw}(G)$. We shall prove that $\mathcal{X}$ satisfies (DPW-5).

Suppose $\mathcal{X}$ does not satisfy (DPW-5) i.e., there exists a quadruple (t_1, t_2, Z_1, Z_2) and $k > 0$ such that $|X_i \cap X_{i+1}| \ge k$ for every $i \in [t_1, t_2 - 1]$[3] and there do not exist k vertex-disjoint directed paths from $Z_2 \subseteq X_{t_2}$ to $Z_1 \subseteq X_{t_1}$ in G. We choose such a quadruple for which $t_2 - t_1$ is minimum. By Menger's theorem there is a separation (A, B) of minimum order $s < k$ that separates (Z_2, Z_1). We construct path decompositions $\mathcal{X}^A$, $\mathcal{X}^B$ of $G[A]$ and $G[B]$ respectively. We concatenate these two path decompositions to obtain a path decomposition $\mathcal{X}'$ of G whose fatness is less than that of $\mathcal{X}$, contradicting our choice of $\mathcal{X}$.

Let $P_1, P_2, \dots, P_s$ be s vertex-disjoint paths from Z_2 to Z_1 such that $q = |\bigcup_{1\le j\le s} V(P_j)|$ is minimum. They exist by the minimality of s. By the minimality of q and (DPW-3'), for each $1 \le j \le s$, the first vertex of P_j is in X_{t_2} and no other vertex of P_j is in X_{t_2}. Similarly, the last vertex of P_j is in X_{t_1} and no other vertex of P_j is in X_{t_1}. By the minimality of $s := |A \cap B|$, for each $j \in [s]$, $|P_j \cap (A \cap B)| = 1$. Let $p_j = P_j \cap (A \cap B)$.

We now construct $\mathcal{X}^B = (X_i^B)_{1\le i\le t_2}$. For each $1 \le i \le t_2$, we define X_i^B as follows:

$$X_i^B = (X_i \cap B) \cup \{ \bigcup_{1\le j\le s} \{p_j : X_i \cap (A \cap P_j) \ne \emptyset\}\}.$$

Claim 8. $\mathcal{X}^B$ is a path decomposition of $G[B]$.

[2](DPW-5) is particularly interesting when $t_1 = t_2$
[3]Hence, $|X_i| \ge k$ for every $i \in [t_1, t_2]$

Proof. We show that $\mathcal{X}^B$ satisfies (DPW-1), (DPW-2) and (DPW-3).

1. It is easy to verify that $\bigcup_{i=1}^{t_2} X_i^B = B$. Hence, (DPW-1) is satisfied.

2. To show that $\mathcal{X}^B$ satisfies (DPW-2) it is enough to show that $\mathcal{X}^B$ satisfies (DPW-2') for $p_j = P_j \cap (A \cap B)$ for any $1 \leq j \leq s$. By construction of $\mathcal{X}_i^B$, it is enough to show that $\{i : X_i \cap (A \cap P_j) \neq \emptyset\}$ is an integer interval in $\mathcal{X}$, for any $1 \leq j \leq s$. This follows from Lemma 6, since for each $1 \leq j \leq s$, the first vertex of P_j is in X_{t_2} and the last vertex of P_j is in X_{t_1}.

3. We know that $\mathcal{X}$ satisfies (DPW-3). Fix j and note that $\{i : p_j \in X_i, 1 \leq i \leq t_2\} \subseteq \{i : p_j \in X_i^B, 1 \leq i \leq t_2\}$. Hence $\mathcal{X}^B$ also satisfies (DPW-3)

□

Similarly we construct $\mathcal{X}^A = (X_i^A)_{t_1 \leq i \leq r}$, a path decomposition of $G[A]$. For each $t_1 \leq i \leq r$, we define X_i^A as follows:

$$X_i^A = (X_i \cap A) \cup \{ \bigcup_{1 \leq j \leq s} \{p_j : X_i \cap (B \cap P_j) \neq \emptyset\}\}.$$

Since $Z_2 \subseteq A$ and $Z_1 \subseteq B$ we have $A \cap B \subseteq X_{t_2}^B \cap X_{t_1}^A$. We concatenate $\mathcal{X}^B$ and $\mathcal{X}^A$ to obtain a path decomposition $\mathcal{X}'$ of G.

Lemma 9. Let $S = A \cap B$. The following are true:

1. For all $1 \leq i \leq r$, if $|X_i^A| = |X_i|$ then $X_i^B \subseteq S$.
2. For all $1 \leq i \leq r$, if $|X_i^B| = |X_i|$ then $X_i^A \subseteq S$.

Proof. We prove (1). The proof of (2) is analogous. We may assume that $t_1 < i < t_2$. Let $|X_i^A| = |X_i|$ and assume that X_i^B is not a subset of S. This means $X_i \cap B \neq \emptyset$. By our construction of X_i^A, for every vertex in $X_i \cap B$ some $p \in A \cap B$ was added in X_i^A. Let $D := X_i^A \setminus X_i$ and $S' := (S \setminus D) \cup (X_i \cap B)$. Note that $|D| = |X_i \cap B|$ and hence, $|S'| = |S|$.

It is easy to see that (i) S' separates (Z_2, Z_1) and (ii) S' separates (Z_2, X_i). This implies that for any $Z \subseteq X_i$ with $|Z| = k$, the quadruple (t, t_2, Z, Z_2) violates (DPW-5). This contradicts the choice of t_1 and t_2. Recall that t_1 and t_2 are chosen to minimize $t_2 - t_1$. □

Lemma 10. There exists $i \in [t_1, t_2]$ such that $|X_i^A| < |X_i|$ and $|X_i^B| < |X_i|$.

Proof. Note that $X_{t_1} \cap B \neq \emptyset$ and $X_{t_2} \cap A \neq \emptyset$. We claim that there exists an $i \in [t_1, t_2]$ such that $X_i \cap A \neq \emptyset$ and $X_i \cap B \neq \emptyset$. Suppose not, since $X_{t_1} \cap B \neq \emptyset$ and $X_{t_2} \cap A \neq \emptyset$, there is an $i \in [t_1, t_2]$ such that $X_i \subseteq B$ and $X_{i+1} \subseteq A$. This implies that $|X_i \cap X_{i+1}| \subseteq A \cap B$. Since $|A \cap B| < k$, i satisfies (DPW-5), which is a contradiction. Hence, there exists an $i \in [t_1, t_2]$ such that $X_i \cap A \neq \emptyset$ and $X_i \cap B \neq \emptyset$. Combining this with Lemma 9, we get the desired lemma. □

Lemma 11. Fatness of $\mathcal{X}'$ is less than that of $\mathcal{X}$.

Proof. Note the following:

- for all $i \in [t_1]$, $X_i^B = X_i$
- for all $i \in [t_1, t_2]$, $|X_i^B| \leq |X_i|$
- for all $i \in [t_2, r]$, $X_i^A = X_i$
- for all $i \in [t_1, t_2]$, $|X_i^A| \leq |X_i|$
- for all $i \in [t_1, t_2]$, $|X_i| \geq k > s$

Hence, by Lemma 9, for every $j > s$ the number of bags of order j in $\mathcal{X}'$ is at most the number of such bags in $\mathcal{X}$. By Lemma 10, there is one such j such that the number of bags of order j in $\mathcal{X}'$ is strictly less than the number of such bags in $\mathcal{X}$. The lemma follows. □

Lemma 11 contradicts our choice of $\mathcal{X}$. Hence, $\mathcal{X}$ satisfies (DPW-5). □

3 Linked DAG decomposition

Definition 12. [DAG-decomposition and DAG-width [BDHK06, Obd06, BDH+12]] A *DAG decomposition* of a digraph G is a pair $\mathcal{D} = (T, \mathcal{X})$ where T is a DAG, and $\mathcal{X} = (X_i)_{i \in V(T)}$ is a family of subsets (node bags) of $V(G)$, such that:

- $\bigcup_{i \in V(T)} X_i = V(G)$. (DGW-1)
- For all nodes $i, j, k \in V(T)$, if $i \preceq j \preceq k$, then $X_i \cap X_k \subseteq X_j$. (DGW-2)
- For all arcs $(i, j) \in E(T)$, $X_i \cap X_j$ guards $X_{\succeq j} \setminus X_i$. For any root $r \in V(T)$, $X_{\succeq r}$ is guarded by $\emptyset$. (DGW-3)

The width of a DAG-decomposition $\mathcal{D} = (T, \mathcal{X})$ is defined as $\max\{|X_i| : i \in V(T)\}$[4]. The *DAG-width* of G, denoted by $\mathsf{dgw}(G)$, is the minimum width over all possible DAG-decompositions of G.

A DAG decomposition is called *linked* if it satisfies the following condition:

- Given any $k > 0$ and $t_1, t_2 \in V(T)$, such that $t_1 \preceq t_2$,
  either G contains k vertex-disjoint directed paths *from* X_{t_2} *to* X_{t_1}
  or there exists $i \in [t_1, t_2]$ such that $|X_i| < k$. (DGW-4)

A DAG decomposition is called *lean* if it satisfies the following condition:

- Given any $k > 0$ and $t_1, t_2 \in V(T)$, such that $t_1 \preceq t_2$,
  and subsets $Z_1 \subseteq X_{t_1}$, and $Z_2 \subseteq X_{t_2}$ such that $|Z_1| = |Z_2| =: k$,
  either G contains k vertex-disjoint directed paths *from* Z_2 *to* Z_1
  or there exists $i \in [t_1, t_2 - 1]$ such that $|X_i \cap X_{i+1}| < k$. (DGW-5)

Lemma 13. (*Path meeting lemma*) *For $l \geq 2$, let $P = v_1 v_2 \ldots v_l$ be a directed path in a digraph G. Let $\mathcal{D} = (T, \mathcal{X})$ be a DAG decomposition of G such that $v_l \in X_a$ and $v_1 \in X_b$ and $a \prec b$. Then $V(P) \cap X_i \neq \emptyset$ for all $a \preceq i \preceq b$.*

[4] Unlike directed pathwidth there is no -1 here.

Proof. Proof is similar to that of Lemma 6. $\square$

Theorem 14. Every digraph G has a DAG decomposition of width $\mathsf{dgw}(G)$ that satisfies (DGW-1) – (DGW-5).

Proof. This proof is a generalization of Theorem 7's proof. Let $\mathcal{D} = (T, \mathcal{X})$ be a DAG decomposition of lexicographically minimal fatness. Suppose $\mathcal{D}$ does not satisfy (DGW-5) i.e., there exists a quadruple (t_1, t_2, Z_1, Z_2) and $k > 0$ such that $|X_i \cap X_{i+1}| \geq k$ for every $i \in [t_1, t_2 - 1]$[5] and there do not exist k vertex-disjoint directed paths from $Z_2 \subseteq X_{t_2}$ to $Z_1 \subseteq X_{t_1}$ in G. We choose such a quadruple for which $d_T(t_1, t_2)$ is minimum. By Menger's theorem there is a separation (A, B) of minimum order $s < k$ that separates (Z_2, Z_1). Let $S = A \cap B$. Among all such separators of minimum order we choose the one that minimizes the following "t_1t_2 distance" of S.

Definition 15. [t_1t_2 distance] Let $S \subseteq V(G)$. The t_1t_2 *distance* of S is defined as $\sum_{v \in S} d_v$, where

$$d_v := \min\{d_T(i, j) \mid i \in [t_1, t_2] \text{ and } v \in X_j\}$$

We construct DAG decompositions $\mathcal{D}^A$, $\mathcal{D}^B$ of $G[A]$ and $G[B]$ respectively. We "merge" these two DAG decompositions to obtain a DAG decomposition $\mathcal{D}'$ of G whose fatness is less than that of $\mathcal{D}$, contradicting our choice of $\mathcal{D}$. Let $P_1, P_2, \ldots, P_s$ be s vertex-disjoint paths from Z_2 to Z_1 such that $q = |\bigcup_{1 \leq j \leq s} V(P_j)|$ is minimum. They exist by the minimality of s. By the minimality of q and (DGW-3), for each $1 \leq j \leq s$, the first vertex of P_j is in X_{t_2} and no other vertex of P_j is in X_{t_2}. Similarly, the last vertex of P_j is in X_{t_1} and no other vertex of P_j is in X_{t_1}. By the minimality of $s := |A \cap B|$, for each $j \in [s]$, $|P_j \cap (A \cap B)| = 1$. Let $p_j = P_j \cap (A \cap B)$.

We now construct $\mathcal{D}^B = (T, \mathcal{X}^B)$, where $\mathcal{X}^B = (X_i^B)_{i \in V(T)}$ is defined as follows:

$$X_i^B = (X_i \cap B) \cup \{\bigcup_{1 \leq j \leq s} \{p_j : X_i \cap (A \cap P_j) \neq \emptyset\}\}.$$

Claim 16. $\mathcal{D}^B$ is a DAG decomposition of $G[B]$.

Proof. We show that $\mathcal{D}^B$ satisfies (DGW-1), (DGW-2) and (DGW-3).

1. It is easy to verify that $\bigcup_{i=1}^{t_2} X_i^B = B$. Hence, (DGW-1) is satisfied.

2. To show that $\mathcal{D}^B$ satisfies (DGW-2) it is enough to show that $\mathcal{D}^B$ satisfies (DGW-2) for $p_j = P_j \cap (A \cap B)$ for any $1 \leq j \leq s$. By construction of $\mathcal{X}_i^B$, it is enough to show that $\{i : X_i \cap (A \cap P_j) \neq \emptyset\}$ is a connected "sub-DAG" of T, for any $1 \leq j \leq s$. This follows from Lemma 13, since for each $1 \leq j \leq s$, the first vertex of P_j is in X_{t_2} and the last vertex of P_j is in X_{t_1}.

3. We know that $\mathcal{D}$ satisfies (DGW-3). Fix j and note that $\{i : p_j \in X_i, 1 \leq i \leq t_2\} \subseteq \{i : p_j \in X_i^B, 1 \leq i \leq t_2\}$. Hence $\mathcal{D}^B$ also satisfies (DGW-3)

$\square$

[5] Hence, $|X_i| \geq k$ for every $i \in [t_1, t_2]$

Similarly we construct $\mathcal{D}^A = (T, \mathcal{X}^A)$, a DAG decomposition of $G[A]$. For each $i \in V(T)$, we define X_i^A as follows:

$$X_i^A = (X_i \cap A) \cup \{ \bigcup_{1 \le j \le s} \{p_j : X_i \cap (B \cap P_j) \neq \emptyset\}\}.$$

Since $Z_2 \subseteq A$ and $Z_1 \subseteq B$ we have $A \cap B \subseteq X_{t_2}^B \cap X_{t_1}^A$. We "merge" $\mathcal{D}^B$ and $\mathcal{D}^A$ by adding a directed edge *from* $X_{t_2}^B$ *to* $X_{t_1}^A$. Let the resulting DAG decomposition be $\mathcal{D}'$.

Lemma 17. Let $S = A \cap B$. The following are true:

1. For all $i \in V(T)$, if $|X_i^A| = |X_i|$ then $X_i^B \subseteq S$.
2. For all $i \in V(T)$, if $|X_i^B| = |X_i|$ then $X_i^A \subseteq S$.

Proof. We prove (1). The proof of (2) is analogous. We may assume that $i \notin V(T) \setminus V(T_{\succ t_1})$ and $i \notin V(T_{\succeq t_2})$. Let $|X_i^A| = |X_i|$ and assume that X_i^B is not a subset of S. This means $X_i \cap B \neq \emptyset$. By our construction of X_i^A, for every vertex in $X_i \cap B$ some $p \in A \cap B$ was added in X_i^A. Let $D := X_i^A \setminus X_i$ and $S' := (S \setminus D) \cup (X_i \cap B)$. Note that $|D| = |X_i \cap B|$ and hence, $|S'| = |S|$.

It is easy to see that (i) S' separates (Z_2, Z_1) and (ii) S' separates (Z_2, X_i). This implies that for any $Z \subseteq X_i$ with $|Z| = k$, the quadruple (t, t_2, Z, Z_2) violates (DGW-5). This contradicts the choice of t_1 and t_2. Recall that t_1 and t_2 are chosen to minimize $d_T(t_1, t_2)$. Hence, $i \notin [t_1, t_2]$.

Let $v \in S \setminus S'$ and $v' \in S' \setminus S$. Let t_v be the "home bag" of v that minimizes the distance d_v (see Definition 15). Consider the underlying DAG T of $\mathcal{D}$. Note that i separates t_v from $T[\{j \in [t_1, t_2]\}]$. Let d be the distance of i from $T[\{j \in [t_1, t_2]\}]$. Since $v' \in X_i$ we have $d_{v'} \le d < d_v$. Hence, the $t_1 t_2$-distance of S' is smaller than that of S, a contradiction. □

Lemma 18. There exists $i \in [t_1, t_2]$ such that $|X_i^A| < |X_i|$ and $|X_i^B| < |X_i|$.

Proof. Note that $X_{t_1} \cap B \neq \emptyset$ and $X_{t_2} \cap A \neq \emptyset$. We claim that there exists an $i \in [t_1, t_2]$ such that $X_i \cap A \neq \emptyset$ and $X_i \cap B \neq \emptyset$. Suppose not, since $X_{t_1} \cap B \neq \emptyset$ and $X_{t_2} \cap A \neq \emptyset$, there is an $i \in [t_1, t_2]$ such that $X_i \subseteq B$ and $X_{i+1} \subseteq A$. This implies that $|X_i \cap X_{i+1}| \subseteq A \cap B$. Since $|A \cap B| < k$, i satisfies (DGW-5), which is a contradiction. Hence, there exists an $i \in [t_1, t_2]$ such that $X_i \cap A \neq \emptyset$ and $X_i \cap B \neq \emptyset$. Combining this with Lemma 9, we get the desired lemma. □

Lemma 19. Fatness of $\mathcal{D}'$ is less than that of $\mathcal{D}$.

Proof. Note the following:

- for all $i \in V(T) \setminus V(T_{\succ t_1})$, $X_i^B = X_i$
- for all $i \in [t_1, t_2]$, $|X_i^B| \le |X_i|$
- for all $i \in V(T_{\succeq t_2})$, $X_i^A = X_i$
- for all $i \in [t_1, t_2]$, $|X_i^A| \le |X_i|$
- for all $i \in [t_1, t_2]$, $|X_i| \ge k > s$

Hence, by Lemma 17, for every $j > s$ the number of bags of order j in $\mathcal{D}'$ is at most the number of such bags in $\mathcal{D}$. By Lemma 18, there is one such j such that the number of bags of order j in $\mathcal{D}'$ is strictly less than the number of such bags in $\mathcal{D}$. The claim follows. □

Lemma 19 contradicts our choice of $\mathcal{D}$. Hence, $\mathcal{D}$ satisfies (DGW-5). □

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